MAT319/320 Ntalampekos/Schul Practice for Midterm 1 Fall 2023

**Write coherent mathematical statements and show your work on all problems. If you use a theorem from the book, you must fully state it. If you give an example then you must prove it is such. Please write clearly.**

## QUESTIONS

(1) Suppose $A, B$ are non-empty subsets of $\mathbb{R}$. Suppose also that for any $a \in A$ and $b \in B$ we have $a < b$.
  (a) Show that $\inf(B) \geq \sup(A)$
  (b) Is it always true that $\inf(B) > \sup(A)$?

(2) State the nested interval property and use part (A) of the previous theorem to prove it.

(3) Without using the nested interval property, show that

$$\bigcap_{n=1}^{\infty} [0, \frac{1}{n}] = \{0\}$$

(4) Show that if $A \subset B \subset \mathbb{R}$ then

$$\inf(B) \leq \inf(A) \leq \sup(A) \leq \sup(B)$$

(5) Let $A \subset \mathbb{R}$ be a set which is above below and is nonempty.
  (a) Recall one definition of supremum for the set $A$.

(6) Denote by $B$ the set

$$B := \{3a : a \in A\}.$$

  Show that

$$\sup(B) = 3 \sup(A).$$

(7) (a) Show that for $A, B \subset \mathbb{R}$ we have

$$\sup(A + B) = \sup(A) + \sup(B)$$

where $A + B := \{a + b : a \in A, b \in B\}$
  (b) If we set $A - B := \{a - b : a \in A, b \in B\}$, is it true that

$$\sup(A - B) = \sup(A) - \sup(B)?$$

(8) Show that $|x - a| < \epsilon$ if and only if $x \in (a - \epsilon, a + \epsilon)$.

(9) Show that if $a \neq b$ are real numbers then there is an $\epsilon > 0$ such that

$$V_\epsilon(a) \cap V_\epsilon(b) = \emptyset$$

(10) Show that there exists a real number $x$ such that $x^2 = 3$.

(11) Show that there is no rational number $r$ such that $r^2 = 3$.

(12) Section 2.4 #11

(13) Section 3.1 #5

(14) Suppose $(x_n)$ is a sequence of real numbers such that $\lim_{n\to\infty} x_n = 3$. Show that for some $N \in \mathbb{N}$ we have that $n > N$ implies $x_n > 1$.

(15) Show that the sequence $a_n = (-1)^n \cdot n^2$ diverges.

(16) Section 3.2, #5

(17) Section 3.2, #7

(18) Section 3.2, #20

(19) Section 3.3 #1

(20) Section 3.3 #7

(21) (10 points) Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $u_1 = 10$ and for all $n \in \mathbb{N}$, $u_{n+1} = 2 + \frac{1}{2}u_n$.
   (a) Show by induction that $u_n \geq 4$ for all $n \in \mathbb{N}$.
   (b) Show that the sequence is non-increasing.

(22) (10 points) Let $(u_n)$ be the sequence from the previous question. You may use what you know from that question. Let

$$a = \inf\{u_n : n \in \mathbb{N}\}.$$

   (a) **Use the definition of limit** to show that the sequence $(u_n)$ is convergent to $a$.
   (b) Find $a$.